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On the Absolute Classification of Quadratic Loci, and on their Intersections with each other and with Linear Loci.

BY WILLIAM E. STORY.

By the absolute classification of loci I mean that classification which is not altered by any real linear transformation, and which is identical with the ordinary classification in so far as the latter is independent of all consideration of the nature of the infinite elements of the loci. Many of the results here obtained are well known, but I believe some of them are new, and the collection of the criteria in such a form that they can be applied to any real forms of the equations of quadratic and linear loci will probably not be without a practical interest. The geometrical conditions are combinations of contact and real and imaginary intersection; the corresponding algebraical criteria can be put into a form in which certain invariants or combinations of the coefficients of covariants are distinguished as zero or positive or negative. A part of this classification has been made in essence by Professor Sylvester in a communication to the Philosophical Magazine, in February, 1851, namely that part which relates to the *contacts of two loci* of the second order, but the form of the investigation is necessarily quite different from that here adopted, where a distinction has to be made between cases in which the relations as to reality as well as contact are different, and where the conditions are required explicitly in terms of the coefficients of the equations of the loci. Homogeneous coordinates are employed throughout the paper; it is assumed that real elements (points, lines and planes) have real coordinates, and that the equations of the loci have real coefficients. Each locus is considered as belonging to a space of the smallest number of dimensions in which it can exist.

I. The equation of a pair of points on a given straight line is

$$U \equiv ax^2 + 2hxy + by^2 = 0,$$

and the discriminant of U is $\Delta = ab - h^2 = (ab)$, say;

if $(ab) > 0$, the points are conjugate imaginary,
if $(ab) < 0$, the points are real and distinct,
if $(ab) = 0$, the points are real and coincident.

II. The equations of a conic and a straight line in a given plane are

$$U \equiv ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0,$$
$$P \equiv \alpha x + \beta y + \gamma z = 0.$$

The intersections of U and P satisfy the equation

$$(a\gamma^2 - 2g\alpha\gamma + c\alpha^2)x^2 + 2(h\gamma^2 - f\alpha\gamma - g\beta\gamma + c\alpha\beta)xy + (b\gamma^2 - 2f\beta\gamma + c\beta^2)y^2 = 0,$$

found by eliminating z between $U = 0$ and $P = 0$, the discriminant of which is

$$\begin{vmatrix} a\gamma^2 - 2g\alpha\gamma + c\alpha^2, & h\gamma^2 - f\alpha\gamma - g\beta\gamma + c\alpha\beta \\ h\gamma^2 - f\alpha\gamma - g\beta\gamma + c\alpha\beta, & b\gamma^2 - 2f\beta\gamma + c\beta^2 \end{vmatrix} = -\gamma^2 \begin{vmatrix} a & h & g & \alpha \\ h & b & f & \beta \\ g & f & c & \gamma \\ \alpha & \beta & \gamma & 0 \end{vmatrix}$$

$= -\gamma^2 \Delta_P = -\gamma^2 (abc)_P$, say; hence, by I,

if $(abc)_P < 0$, U and P intersect in imaginary points,
if $(abc)_P > 0$, U and P intersect in distinct real points,
if $(abc)_P = 0$, U and P intersect in coincident real points.

If $\gamma = 0$ the above discriminant vanishes identically, but eliminating x or y instead of z, the discriminant of the resultant equation is $-\alpha^2\Delta_P$ or $-\beta^2\Delta_P$, and the criteria are the same as those just given.

$\Delta_P = 0$ is in general the condition that P is tangent to U (*i. e.* is the tangential equation of U), but if U consists of two straight lines, $\Delta_P = 0$ is the condition that P passes through the double point of U.

III. The nature of the conic U, where

$$U \equiv ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy,$$

is found by considering its intersections with a real line rotating about a fixed point, namely: if the intersections are imaginary for all positions of the rotating line, U is an imaginary true conic; if the intersections are imaginary for all positions but one, U consists of a pair of conjugate imaginary straight lines whose intersection lies on the rotating line in that one position; if the intersections are real for an infinite succession of positions, and imaginary for an infinite succession, U is a real true conic; and if the intersections are real for all positions, U is either a real true conic or consists of a pair of real or imaginary straight lines.

A real cone is met by any edge of the tetrahedron in imaginary points, in distinct real points, or in coincident real points so that the edge is tangent to the cone or passes through the vertex, or the cone contains the edge as a generator; the intersections with the revolving plane are of the species 5, 4, 6, 13, 11 or 12, namely, of the species 12 for at most three edges meeting in one point. Now by a real linear transformation which does not alter the value of Δ, any tangent to the cone can be made an edge of the tetrahedron of reference, hence $\Delta = 0$. Then the conditions are evidently

$$(abcd) = 0,\ \text{either } (ab) \overset{=}{<} 0,\ \text{or } a(abc) \overset{=}{<} 0 \text{ and } a(abd) \overset{=}{<} 0,$$

and either $(cd) \overset{=}{<} 0$, or $c(acd) \overset{=}{<} 0$, and $c(bcd) \overset{=}{<} 0$.

A pair of imaginary planes is met by any edge of the tetrahedron in two imaginary points, or in two coincident real points, or the edge lies entirely in the quadric (is the double line of the pair of planes), but only one edge can have the latter position; the intersections with the revolving plane are of the species 9, 10 or 14, namely, of the species 9 for at least one edge, and of the species 14 for at most one edge. The conditions are therefore

$$(abcd) = 0,\ (bcd) = (acd) = (abd) = (abc) = 0,\ \text{and either } (ab) > 0$$
$$\text{or } (ac) > 0 \text{ or } (ad) > 0 \text{ or } (bc) > 0 \text{ or } (bd) > 0 \text{ or } (cd) > 0.$$

A pair of distinct real planes is met by any edge of the tetrahedron in two distinct real points, or in two coincident real points, or the edge lies in one or both of the planes of the pair, and at least one edge must have the first position and at most one edge can have the last position (coincident with the double line of the pair of planes); the intersections with the revolving plane are of the species 11, 12 or 14, namely, of the species 11 for at least one edge, and of the species 14 for at most one edge. By a real linear transformation which does not alter the value of Δ, the double line of the pair of planes can be made an edge of the tetrahedron, hence $\Delta = 0$. The conditions are then

$$(abcd) = 0,\ (bcd) = (acd) = (abd) = (abc) = 0,\ \text{and either } (ab) < 0$$
$$\text{or } (ac) < 0 \text{ or } (ad) < 0 \text{ or } (bc) < 0 \text{ or } (bd) < 0 \text{ or } (cd) < 0.$$

A pair of coincident real planes is met by any edge of the tetrahedron in two coincident real points; the intersections with the revolving plane are of the species 14 for every edge. The conditions are therefore

$$(abcd) = 0,\ (bcd) = (acd) = (abd) = (abc) = 0,$$
$$(ab) = (ac) = (ad) = (bc) = (bd) = (cd) = 0.$$

The conditions here given for the different classes are mutually exclusive and therefore not only necessary but sufficient. They can be put into a somewhat

$h \lessgtr 0$ the upper sign must be given to f, and $U \equiv (ax + by + gz)^2$, *i. e.* U consists of two coincident real lines; if $a \lessgtr 0$ and $h = 0$, then $b = f = 0$, and $U \equiv ax^2 + 2gzx + cz^2$, where $ac - g^2 = 0$, *i. e.* U consists of two coincident real lines; similarly if $a \lessgtr 0$ and $g = 0$; if $a = 0$, and either $b \lessgtr 0$ or $c \lessgtr 0$, then $g = h = 0$, $U \equiv by^2 + 2fyz + cz^2$, where $bc - f^2 = 0$, *i. e.* U still consists of two coincident real lines; if $a = b = c = 0$, then $f = g = h = 0$ and there is no conic.

The criteria for the conic U are then the following:

$(abc) \lessgtr 0$, $a\,(abc) > 0$, $(ab) > 0$, U is an imaginary true conic;

$(abc) \lessgtr 0$, and either $a\,(abc) \overset{=}{<} 0$ or $(ab) \overset{=}{<} 0$, U is a real true conic;

$(abc) = 0$, and either $(bc) > 0$ or $(ac) > 0$ or $(ab) > 0$, U is a pair of imaginary lines;

$(abc) = 0$, and either $(bc) < 0$ or $(ac) < 0$ or $(ab) < 0$, U is a pair of distinct real lines;

$(abc) = 0$, and $(bc) = (ac) = (ab) = 0$, U is a pair of coincident real lines.

IV. If U and V are two conics, where

$$U \equiv ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy.$$
$$V \equiv a'x^2 + b'y^2 + c'z^2 + 2f'yz + 2g'zx + 2h'xy,$$

among the conics $\lambda U + \mu V = 0$ passing through the intersections of U and V are three pairs of lines for which $\lambda : \mu$ is determined by the equation $\Delta_{\lambda\mu} = 0$, where

$$\Delta_{\lambda\mu} \equiv \begin{vmatrix} \lambda a + \mu a', & \lambda h + \mu h', & \lambda g + \mu g' \\ \lambda h + \mu h', & \lambda b + \mu b', & \lambda f + \mu f' \\ \lambda g + \mu g', & \lambda f + \mu f', & \lambda c + \mu c' \end{vmatrix};$$

if the intersections of U, V are all real, the three pairs of lines are real; if two of the intersections of U, V are real and two imaginary, the lines of one pair are real and those of each of the two other pairs are non-conjugate imaginary; if the intersections of U, V are all imaginary, the lines of one pair are real and those of each of the other pairs are conjugate imaginary. Hence if the four intersections of U, V are all real or all imaginary the three roots of $\Delta_{\lambda\mu} = 0$ are real, but if U and V have two real and two imaginary intersections, one of the roots of $\Delta_{\lambda\mu} = 0$ is real and two are (of course conjugate) imaginary. Putting then* $R = \Theta^2\Theta'^2 + 18\Delta\Delta'\Theta\Theta' - 27\Delta^2\Delta'^2 - 4\Delta\Theta'^3 - 4\Delta'\Theta^3$,

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* Salmon's Conic Sections, § 372.

$\Delta_{\lambda\mu} = 0$ will have three distinct real roots if $R > 0$, two imaginary roots and one real root if $R < 0$, and three real roots of which two at least are equal if $R = 0$.

If $R < 0$, U and V have two distinct real intersections and two imaginary intersections.

If $R > 0$, U and V have four distinct real intersections or four distinct imaginary intersections, and it remains to distinguish between these two cases.* For then the roots of $\Delta_{\lambda\mu} = 0$ are all real and distinct, and hence the reduction to the common polar triangle is real, *i. e.* by a real linear transformation U and V can be brought at once to the forms

$$U_0 \equiv \alpha x_0^2 + \beta y_0^2 + \gamma z_0^2, \quad V_0 \equiv \alpha' x_0^2 + \beta' y_0^2 + \gamma' z_0^2.$$

Now the intersections will be all imaginary if an imaginary conic whose equation is real can be found passing through these intersections, *i. e.* if $\lambda U + \mu V = 0$ represents an imaginary conic for any real values of λ, μ; by the criteria of III this will happen for any values of λ, μ, if $\lambda\alpha + \mu\alpha'$, $\lambda\beta + \mu\beta'$, $\lambda\gamma + \mu\gamma'$ have the same sign, *i. e.* if λ, μ, ρ, p, q, r can be determined as real quantities such that

$$\lambda\alpha + \mu\alpha' = \rho p^2, \quad \lambda\beta + \mu\beta' = \rho q^2, \quad \lambda\gamma + \mu\gamma' = \rho r^2,$$

i. e. if real quantities p, q, r can be so determined that

$$\begin{vmatrix} \alpha & \alpha' & p^2 \\ \beta & \beta' & q^2 \\ \gamma & \gamma' & r^2 \end{vmatrix} = (\beta\gamma' - \beta'\gamma)\, p^2 + (\gamma\alpha' - \gamma'\alpha)\, q^2 + (\alpha\beta' - \alpha'\beta)\, r^2 = 0,$$

which will be possible if $(\beta\gamma' - \beta'\gamma)$, $(\gamma\alpha' - \gamma'\alpha)$, $(\alpha\beta' - \alpha'\beta)$ are not all of the same sign, and only then, *i. e.* U and V will have four real intersections if $(\beta\gamma' - \beta'\gamma)$, $(\gamma\alpha' - \gamma'\alpha)$, $(\alpha\beta' - \alpha'\beta)$ are all of the same sign. Now let $\mathbf{F}$ be the quadratic covariant of U and V, *i. e.*†

$$\begin{aligned} \mathbf{F} \equiv {} & (BC' + B'C - 2FF')\, x^2 + (CA' + C'A - 2GG')\, y^2 + (AB' + A'B - 2HH')\, z^2 \\ & + 2(GH' + G'H - AF' - A'F)\, yz + 2(HF' + H'F - BG' - B'G)\, zx \\ & + 2(FG' + F'G - CH' - C'H)\, xy, \end{aligned}$$

where A, B, C, F, G, H and A', B', C', F', G', H' are the first minors of Δ and Δ', the discriminants of U and V respectively. This covariant formed for U_0 and V_0 is

$$\mathbf{F}_0 \equiv \alpha\alpha'(\beta\gamma' + \gamma\beta')\, x_0^2 + \beta\beta'(\gamma\alpha' + \alpha\gamma')\, y_0^2 + \gamma\gamma'(\alpha\beta' + \beta\alpha')\, z_0^2;$$

the roots of $\Delta_{\lambda\mu} = 0$ formed for U, V, and for U_0, V_0 are evidently the same, and if the coefficients α, β, γ, α', β', γ' are so taken in absolute value that the two equations $\Delta_{\lambda\mu} = 0$ are not simply equivalent but identical, *i. e.* if the determinant of substitution is unity, then, as in Salmon's Geometry of Three Dimensions, §§ 215, 234,

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* Salmon says they "have not been distinguished by any simple criterion."

† Salmon's Conic Sections, § 378.

$$\Delta = \alpha\beta\gamma;\ \Delta' = \alpha'\beta'\gamma',\ U \equiv U_0,\ V \equiv V_0,\ \mathbf{F} \equiv \mathbf{F}_0,$$

which gives the three equation for x_0^2, y_0^2, z_0^2,

$$\alpha x_0^2 + \beta y_0^2 + \gamma z_0^2 \equiv U,$$
$$\alpha' x_0^2 + \beta' y_0^2 + \gamma' z_0^2 \equiv V,$$
$$\alpha\alpha'(\beta\gamma' + \gamma\beta')x_0^2 + \beta\beta'(\gamma\alpha' + \alpha\gamma')y_0^2 + \gamma\gamma'(\alpha\beta' + \beta\alpha')z_0^2 \equiv \mathbf{F},$$

from which follow

$$x_0^2 : y_0^2 : z_0^2 \equiv \begin{vmatrix} U, & \beta, & \gamma \\ V, & \beta', & \gamma' \\ \mathbf{F}, & \beta\beta'(\gamma\alpha' + \alpha\gamma'), & \gamma\gamma'(\alpha\beta' + \beta\alpha') \end{vmatrix} : \text{etc.}$$
$$\equiv (\beta\gamma' - \gamma\beta')(-\alpha\beta'\gamma' U - \alpha'\beta\gamma V + \mathbf{F}) : \text{etc.}$$
$$\equiv (\beta\gamma' - \gamma\beta')\left(-\frac{\alpha}{\alpha'}\Delta' U - \frac{\alpha'}{\alpha}\Delta V + \mathbf{F}\right) : \text{etc.}$$

or, introducing the roots of $\Delta_{\lambda\mu} = 0$, $\frac{\lambda_1}{\mu_1} = -\frac{\alpha'}{\alpha}$, $\frac{\lambda_2}{\mu_2} = -\frac{\beta'}{\beta}$, $\frac{\lambda_3}{\mu_3} = -\frac{\gamma'}{\gamma}$, and putting

$$\frac{\mu_1}{\lambda_1}\Delta' U + \frac{\lambda_1}{\mu_1}\Delta V + \mathbf{F} \equiv X,\ \frac{\mu_2}{\lambda_2}\Delta' U + \frac{\lambda_2}{\mu_2}\Delta V + \mathbf{F} \equiv Y,\ \frac{\mu_3}{\lambda_3}\Delta' U + \frac{\lambda_3}{\mu_3}\Delta V + \mathbf{F} \equiv Z,$$

we have $\qquad x_0^2 : y_0^2 : z_0^2 \equiv (\beta\gamma' - \gamma\beta')X : (\gamma\alpha' - \alpha\gamma')Y : (\alpha\beta' - \beta\alpha')Z,$

from which it is evident that $(\beta\gamma' - \gamma\beta')$, $(\gamma\alpha' - \alpha\gamma')$, $(\alpha\beta' - \beta\alpha')$ have respectively the signs of X, Y, Z, or their opposites, *i. e.* the intersections of U, V will be all four real if the three last mentioned quantities have the same signs, and only then. But X, Y, Z are real, since the roots of $\Delta_{\lambda\mu} = 0$ are real, hence they will have the same sign if their sum s_1 and product s_3 have the same sign and the sum of their binary products s_2 is positive.

Let σ_1 be the sum, σ_3 the product, σ_2 the sum of binary products of $\frac{\lambda_1}{\mu_1}$, $\frac{\lambda_2}{\mu_2}$, $\frac{\lambda_3}{\mu_3}$, then we have, writing $\Delta_{\lambda\mu} \equiv \Delta.\lambda^3 + \Theta.\lambda^2\mu + \Theta'.\lambda\mu^2 + \Delta'.\mu^3$,

$$\Delta.\sigma_1 = -\Theta,\ \Delta.\sigma_2 = \Theta',\ \Delta.\sigma_3 = -\Delta',$$

$$s_1 = \frac{\sigma_2}{\sigma_3}\Delta'.U + \sigma_1\Delta.V + 3\mathbf{F} = -\Theta'.U - \Theta.V + 3\mathbf{F},$$

$$\begin{aligned} s_2 &= \frac{\sigma_1}{\sigma_3}\Delta'^2.U^2 + \frac{1}{3}\frac{\sigma_1\sigma_2 - 3\sigma_3}{\sigma_3}\Delta\Delta'.UV + \sigma_2\Delta^2.V^2 + 2\frac{\sigma_2}{\sigma_3}\Delta'.U\mathbf{F} \\ &\quad + 2\sigma_1\Delta.V\mathbf{F} + 3\mathbf{F}^2 \\ &= \Delta'\Theta.U^2 + \tfrac{1}{3}(\Theta\Theta' - 3\Delta\Delta')UV + \Delta\Theta'.V^2 - 2\Theta'.U\mathbf{F} - 2\Theta.V\mathbf{F} + 3\mathbf{F}^2, \end{aligned}$$

$$\begin{aligned}s_3 = {} & \frac{1}{\sigma_3}\Delta'^3.U^3 + \frac{\sigma_1^2 - 2\sigma_2}{\sigma_3}\Delta\Delta'^2.U^2V + \frac{\sigma_2^2 - 2\sigma_1\sigma_3}{\sigma_3}\Delta^2\Delta'.UV^2 + \sigma_3\Delta^3.V^3 \\ & + \frac{\sigma_1}{\sigma_3}\Delta'^2.U^2\mathbf{F} + \frac{1}{3}\frac{\sigma_1\sigma_2 - 3\sigma_3}{\sigma_3}\Delta\Delta'.UV\mathbf{F} + \sigma_2\Delta^2.V^2\mathbf{F} + \frac{\sigma_2}{\sigma_3}\Delta'.U\mathbf{F}^2 \\ & + \sigma_1\Delta.V\mathbf{F}^2 + \mathbf{F}^3 = -\Delta\Delta'^2.U^3 + (2\Delta\Theta' - \Theta^2)\Delta'.U^2V \\ & + (2\Delta'.\Theta - \Theta'^2)\Delta.UV^2 - \Delta^2\Delta'.V^3 + \Delta'\Theta.U^2\mathbf{F} + \tfrac{1}{3}(\Theta\Theta' - 3\Delta\Delta').UV\mathbf{F} \\ & + \Delta\Theta'.V^2\mathbf{F} - \Theta'.U\mathbf{F}^2 - \Theta.V\mathbf{F}^2 + \mathbf{F}^3,\end{aligned}$$

where*

$$\Theta = Aa' + Bb' + Cc' + 2Ff' + 2Gg' + 2Hh',$$
$$\Theta' = A'a + B'b + C'c + 2F'f + 2G'g + 2H'h.$$

If then $s_1 s_3 \geqq 0$ and $s_2 > 0$ for all values of the variables, U and V have four distinct real intersections, otherwise they have four distinct imaginary intersections. It may be remarked that we cannot have $s_1 = 0$ if $s_2 > 0$, and $s_3 = 0$ only for those values of the variables for which $x_0 = 0$, $y_0 = 0$, or $z_0 = 0$, *i. e.* for points on the sides of the common self-conjugate triangle.

If $R = 0$, U and V have contact, *i. e.* two of their intersections are coincident. If the other two are distinct from each other and from the double intersection, this is real and the distinct intersections may be real or conjugate imaginary; in the former case the three pairs of lines are real and two of them coincide, in the latter case the lines of one pair are real and those of the other pairs (which coincide) conjugate imaginary. In either case $\Delta_{\lambda\mu} = 0$ has three real roots, two of which are coincident. These cases may be distinguished by a method similar to that employed above for $R > 0$, namely, U and V cannot be put into the canonical form before used, but they can be brought by a real linear transformation into the forms†

$$U_0 \equiv \alpha x_0^2 + \beta y_0^2 + 2\gamma z_0 x_0, \quad V_0 \equiv \alpha' x_0^2 + \beta' y_0^2 + 2\gamma' z_0 x_0.$$

The equation $\Delta_{\lambda\mu} = 0$ formed for U_0, V_0 is

$$0 = \begin{vmatrix} \lambda\alpha + \mu\alpha', & 0, & \lambda\gamma + \mu\gamma' \\ 0, & \lambda\beta + \mu\beta', & 0 \\ \lambda\gamma + \mu\gamma', & 0, & 0 \end{vmatrix} = -(\lambda\beta + \mu\beta')(\lambda\gamma + \mu\gamma')^2,$$

i. e. if $\frac{\lambda_1}{\mu_1}$ is the double and $\frac{\lambda_2}{\mu_2}$ the single root, $\frac{\lambda_1}{\mu_1} = -\frac{\gamma'}{\gamma}$, $\frac{\lambda_2}{\mu_2} = -\frac{\beta'}{\beta}$. The pairs of straight lines through the intersections are then

$$(\gamma\alpha' - \alpha\gamma')x_0^2 - (\beta\gamma' - \gamma\beta')y_0^2 = 0 \text{ and } (\alpha\beta' - \beta\alpha')x_0^2 - 2(\beta\gamma' - \gamma\beta')z_0x_0 = 0,$$

and the former is the double pair. This double pair will be real or imaginary according as $\beta\gamma' - \gamma\beta'$ and $\gamma\alpha' - \alpha\gamma'$ are of like or unlike signs, *i. e.* the distinct intersections of U, V will be real or imaginary in these cases respectively. Now the covariant $\mathbf{F}$ formed for U_0, V_0 is

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*Salmon's Conic Sections, § 370.

† Clebsch's Vorlesungen über Geometrie, I, p. 137.

$$\mathbf{F}_0 \equiv -(\alpha\beta\gamma'^2 + \gamma^2\alpha'\beta')\,x_0^2 - 2\beta\gamma\beta'\gamma'\,y_0^2 - 2\gamma\gamma'(\beta\gamma' + \gamma\beta')\,z_0x_0,$$

and, if the determinant of substitution for passing from U, V to U_0, V_0 is unity,

$$\Delta = -\beta\gamma^2,\ \ \Delta' = -\beta'\gamma'^2,$$
$$\alpha x_0^2 + \beta y_0^2 + 2\gamma z_0x_0 \equiv U,$$
$$\alpha' x_0^2 + \beta' y_0^2 + 2\gamma' z_0x_0 \equiv V,$$
$$-(\alpha\beta\gamma'^2 + \gamma^2\alpha'\beta')\,x_0^2 - 2\beta\gamma\beta\gamma' y_0^2 - 2\gamma\gamma'(\beta\gamma' + \gamma\beta')\,z_0x_0 \equiv \mathbf{F},$$

from which follows $\left(\text{observing that here } \frac{\lambda_3}{\mu_3} = \frac{\lambda_1}{\mu_1}\right)$,

$$X \equiv Z \equiv \frac{\mu_1}{\lambda_1}\Delta'.U + \frac{\lambda_1}{\mu_1}\Delta.V + \mathbf{F} \equiv (\beta\gamma' - \gamma\beta')(\gamma\alpha' - \alpha\gamma')\,x_0^2,$$
$$Y \equiv \frac{\mu_2}{\lambda_2}\Delta'.U + \frac{\lambda_2}{\mu_2}\Delta.V + \mathbf{F} \equiv (\beta\gamma' - \gamma\beta')^2\,y_0^2;$$

hence $Y \geqq 0$ and $X \equiv Z \geqq 0$ or $\leqq 0$ according as $(\beta\gamma' - \gamma\beta')$ and $(\gamma\alpha' - \alpha\gamma')$ have like or unlike signs, *i. e.* according as the distinct intersections of U and V are real or imaginary; also, using s_1, s_2, s_3 as above,

$$s_1 \equiv (\beta\gamma' - \gamma\beta')[2(\gamma\alpha' - \alpha\gamma')\,x_0^2 + (\beta\gamma' - \gamma\beta')\,y_0^2],$$
$$s_2 \equiv (\beta\gamma' - \gamma\beta')^2(\gamma\alpha' - \alpha\gamma')[(\gamma\alpha' - \alpha\gamma')\,x_0^2 + 2(\beta\gamma' - \gamma\beta')\,y_0^2]\,x_0^2,$$
$$s_3 \equiv (\beta\gamma' - \gamma\beta')^4(\gamma\alpha' - \alpha\gamma')^2\,x_0^4y_0^2,$$

i. e. $s_1 \geqq 0$, $s_2 \geqq 0$, $s_3 \geqq 0$ for all real values of the variables if the distinct intersections of U and V are real, and only then ($s_1 = 0$ only for the point of contact, $s_2 = 0$ only for a point on the double common tangent, and $s_3 = 0$ only for a point on the double common tangent or on the common polar with respect to U and V of the intersection of that tangent with the common chord joining the distinct intersections); *i. e.* (noticing that $s_3 \geqq 0$ for all real values of the variables) if any set of real values of the variables exists for which either $s_1 < 0$ or $s_2 < 0$, the distinct intersections of U and V are imaginary.

If U and V have two distinct pairs of coincident intersections, *i. e.* if they have two distinct contacts, we may employ the canonical forms*

$$U \equiv \beta y_0^2 + 2\alpha z_0x_0,\ \ V \equiv \beta' y_0^2 + 2\alpha' z_0x_0;$$

then $\quad \Delta = -\alpha^2\beta,\ \Delta' = -\alpha'^2\beta',\ \Delta_{\lambda\mu} \equiv -(\lambda\alpha + \mu\alpha')^2(\lambda\beta + \mu\beta'),\ R = 0,$

$$\frac{\lambda_1}{\mu_1} = \frac{\lambda_3}{\mu_3} = -\frac{\alpha'}{\alpha},\ \ \frac{\lambda_2}{\mu_2} = -\frac{\beta'}{\beta},\ \ \alpha\beta' - \beta\alpha' \gtrless 0,$$

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* Clebsch, Vorlesungen über Geometrie, I, p. 139.

and $\alpha' U - \alpha V \equiv \lambda_1 U + \mu_1 V = 0$ represents a double straight line (the chord of contact), while $\beta' U - \beta V \equiv \lambda_2 U + \mu_2 V = 0$ represents the tangents at the contacts, *i. e.* a pair of distinct straight lines, real or imaginary according as the two contacts are real or imaginary. Now in this case

$$\mathbf{F} \equiv -2\alpha\alpha' [\beta\beta' y_0^2 + (\alpha\beta' + \beta\alpha') z_0 x_0] \equiv -\alpha\alpha' (\beta' U + \beta V),$$

and, using $X \equiv Z$ and Y as before,

$$X \equiv \frac{\mu_1}{\lambda_1} \Delta'. U + \frac{\lambda_1}{\mu_1} \Delta. V + \mathbf{F} \equiv \alpha\alpha' (\beta' U + \beta V) - \alpha\alpha' (\beta' U + \beta V) \equiv 0,$$

$$Y \equiv \frac{\mu_2}{\lambda_2} \Delta'. U + \frac{\lambda_2}{\mu_2} \Delta. V + \mathbf{F} \equiv \beta\alpha'^2. U + \alpha^2\beta'. V - \alpha\alpha' (\beta' U + \beta V)$$

$$\equiv (\alpha\beta' - \beta\alpha')(\alpha' U - \alpha V) \equiv -(\alpha\beta' - \beta\alpha')^2 y_0^2,$$

$$s_1 \equiv 2X + Y \equiv Y, \quad s_2 \equiv X^2 + 2XY \equiv 0, \quad s_3 \equiv X^2 Y \equiv 0;$$

so that for all values of the variables $s_1 \equiv Y \leqq 0$. The double root of

$$\Delta_{\lambda\mu} \equiv \Delta. \lambda^3 + \Theta. \lambda^2\mu + \Theta'. \lambda\mu^2 + \Delta'. \mu^3 = 0$$

is given by $$\frac{\lambda_1}{\mu_1} = -\frac{2(3\Delta'\Theta - \Theta'^2)}{9\Delta\Delta' - \Theta\Theta'} = -\frac{9\Delta\Delta' - \Theta\Theta'}{2(3\Delta\Theta' - \Theta^2)},$$

so that $$\frac{\lambda_1^2}{\mu_1^2} = \frac{3\Delta'\Theta - \Theta'^2}{3\Delta\Theta' - \Theta^2},$$

and the distinct root is given by

$$\frac{\lambda_2}{\mu_2} = -\frac{\Delta'\mu_1^2}{\Delta\lambda_1^2} = -\frac{\Delta'(3\Delta\Theta' - \Theta^2)}{\Delta(3\Delta'\Theta - \Theta'^2)},$$

and therefore the contacts will be real or imaginary according as

$$\Delta'(3\Delta\Theta' - \Theta^2) U - \Delta(3\Delta'\Theta - \Theta'^2) V = 0$$

represents a pair of real or imaginary straight lines, *i. e.*, by the criteria obtained in III, according as those of the three quantities

$$L = \Delta'^2(3\Delta\Theta' - \Theta^2)^2 A - \Delta\Delta'(3\Delta\Theta' - \Theta^2)(3\Delta'\Theta - \Theta'^2)(bc' + cb' - 2ff') + \Delta^2(3\Delta'\Theta - \Theta'^2)^2 A',$$

$$M = \Delta'^2(3\Delta\Theta' - \Theta^2)^2 B - \Delta\Delta'(3\Delta\Theta' - \Theta^2)(3\Delta'\Theta - \Theta'^2)(ca' + ac' - 2gg') + \Delta^2(3\Delta'\Theta - \Theta'^2)^2 B',$$

$$N = \Delta'^2(3\Delta\Theta' - \Theta^2)^2 C - \Delta\Delta'(3\Delta\Theta' - \Theta^2)(3\Delta'\Theta - \Theta'^2)(ab' + ba' - 2hh') + \Delta^2(3\Delta'\Theta - \Theta'^2)^2 C'$$

which do not vanish are negative or positive (namely, we have proved above that such of them as do not vanish are of the same sign, and evidently they will all vanish only when the tangents at the contacts coincide, *i. e.* when U and V have four-point contact).

If three of the intersections of U and V coincide, *i. e.* if U and V have three-point contact, the three roots of $\Delta_{\lambda\mu} = 0$ are equal, *i. e.*

$$\frac{3\Delta'}{\Theta'} = \frac{\Theta'}{\Theta} = \frac{\Theta}{3\Delta} = \sqrt[3]{\frac{\Delta'}{\Delta}} = -\frac{\lambda_1}{\mu_1} = -\frac{\lambda_2}{\mu_2} = -\frac{\lambda_3}{\mu_3}$$

(and of course $R=0$), and $\lambda_1 U+\mu_1 V=0$, *i. e.* $\sqrt[3]{\Delta'}\,U-\sqrt[3]{\Delta}\,V=0$, where the real cube roots are to be taken, represents a pair of lines consisting of the common tangent at the triple intersection and the line joining this point with the isolated intersection. The equation of these lines may be written in either of the forms $3\Delta' U-\Theta' V=0$, $\Theta' U-\Theta V=0$ or $\Theta U-3\Delta V=0$, but either of these forms becomes illusory in some case (namely, the first two if $\Delta'=0$, and the last two if $\Delta=0$, for then $\Theta=0$ and $\Theta'=0$), but the form above used is always definite. In this case the canonical forms

$$U\equiv\beta y_0^2+2\gamma z_0 x_0,\quad V\equiv\rho\beta y_0^2+2\rho\gamma z_0 x_0+2\alpha' x_0 y_0$$

may be employed; then

$$\Delta=-\beta\gamma^2,\quad \Delta'\equiv-\rho^3\beta\gamma^2,\quad \Delta_{\lambda\mu}\equiv-\beta\gamma^2(\lambda+\mu\rho)^3,\quad \frac{\lambda_1}{\mu_1}=-\rho,$$

$$\begin{aligned}\mathbf{F}&\equiv\gamma^2\left[\alpha'^2x_0^2-2\rho^2\beta^2y_0^2-4\rho^2\beta\gamma z_0x_0-2\rho\beta\alpha' x_0y_0\right]\\ &\equiv\gamma^2\left[\alpha'^2x_0^2-\rho\beta(\rho U+V)\right],\end{aligned}$$

$$X\equiv Y\equiv Z\equiv\rho\beta\gamma^2(\rho U+V)+\mathbf{F}\equiv\gamma^2\alpha'^2x_0^2,$$

$$s_1\equiv3\gamma^2\alpha'^2x_0^2,\quad s_2\equiv3\gamma^4\alpha'^4x_0^4,\quad s_3\equiv\gamma^6\alpha'^6x_0^6,$$

i. e. $s_1\overset{=}{>}0$, $s_2\overset{=}{>}0$, $s_3\overset{=}{>}0$.

If all four intersections of U and V coincide, *i. e.* if U and V have four-point contact, the three roots of $\Delta_{\lambda\mu}=0$ are equal, *i. e.*

$$\frac{3\varDelta'}{\Theta'}=\frac{\Theta'}{\Theta}=\frac{\Theta}{3\varDelta}=\sqrt[3]{\frac{\varDelta'}{\varDelta}}=-\frac{\lambda_1}{\mu_1}=-\frac{\lambda_2}{\mu_2}=-\frac{\lambda_3}{\mu_3}$$

(and of course $R=0$), and $\sqrt[3]{\Delta'}\,U-\sqrt[3]{\Delta}\,V=0$ represents double the tangent at the quadruple intersection, *i. e.* by III,

$$\sqrt[3]{\Delta'^2}\,A-\sqrt[3]{\Delta\Delta'}\,(bc'+cb'-2ff')+\sqrt[3]{\Delta^2}\,A'=0,$$

$$\sqrt[3]{\Delta'^2}\,B-\sqrt[3]{\Delta\Delta'}\,(ca'+ac'-2gg')+\sqrt[3]{\Delta^2}\,B'=0,$$

$$\sqrt[3]{\Delta'^2}\,C-\sqrt[3]{\Delta\Delta'}\,(ab'+ba'-2hh')+\sqrt[3]{\Delta^2}\,C'=0.$$

Convenient canonical forms of U and V are*

$$U\equiv\alpha x_0^2+\beta y_0^2+2\gamma x_0y_0+2\delta z_0x_0,\quad V\equiv\rho\alpha x_0^2+\sigma(\beta y_0^2+2\gamma x_0y_0+2\delta z_0x_0);$$

then

$$\Delta=-\beta\delta^2,\quad \Delta'=-\sigma^3\beta\delta^2,\quad \Delta_{\lambda\mu}\equiv-\beta\delta^2(\lambda+\mu\sigma)^3,\quad \frac{\lambda_1}{\mu_1}=-\sigma,$$

$$\begin{aligned}\mathbf{F}&\equiv-\sigma\beta\delta^2\left[(\rho+\sigma)\alpha x_0^2+2\sigma\beta y_0^2+4\sigma\delta z_0x_0+4\sigma\gamma x_0y_0\right]\\ &\equiv-\sigma\beta\delta^2(\sigma U+V),\end{aligned}$$

$$X\equiv Y\equiv Z\equiv\sigma\beta\delta^2(\sigma U+V)+\mathbf{F}\equiv0,$$

$$s_1\equiv0,\quad s_2\equiv0,\quad s_3\equiv0.$$

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* Clebsch, Geometrie, p. 141.

If U and V have a straight line in common, the conic $\lambda U + \mu V$ contains this line, whatever the values of λ, μ; hence $\Delta_{\lambda\mu} \equiv 0$, *i. e.*

$$\Delta = 0, \quad \Theta = 0, \quad \Theta' = 0, \quad \Delta' = 0.$$

If the second lines of U and V are distinct from each other and from the first line, and the double points are also distinct, we may write

$$U \equiv 2\beta x_0 y_0, \quad V \equiv 2\gamma' x_0 z_0, \quad \mathbf{F} \equiv \beta^2 \gamma'^2 x_0^2,$$

i. e. $\mathbf{F} \overset{=}{>} 0$ for all real values of the variables.

If the second lines of U and V are distinct from each other and from the first line, but the double points coincident, we may write

$$U \equiv 2\beta x_0 y_0, \quad V \equiv \alpha' x_0^2 + 2\beta' x_0 y_0, \quad \mathbf{F} \equiv 0,$$

but, since the lines of each pair are distinct and real, either $A < 0$ or $B < 0$ or $C < 0$, and either $A' < 0$ or $B' < 0$ or $C' < 0$.

If the second line of U coincides with the line common to U and V, we may write $\qquad U \equiv \alpha x_0^2, \quad V \equiv 2\beta' x_0 y_0, \quad \mathbf{F} \equiv 0,$

and then $\qquad A = B = C = 0$, and either $A' < 0$ or $B' < 0$ or $C' < 0$.

Finally, if U and V are identical,

$$\frac{a'}{a} = \frac{b'}{b} = \frac{c'}{c} = \frac{f'}{f} = \frac{g'}{g} = \frac{h'}{h}.$$

Collecting these results, the criteria for the intersections of the two conics U and V are as follows:

1. $R < 0$; two real and distinct and two conjugate imaginary intersections.
2. $R > 0$, $s_2 > 0$, $s_1 s_3 \overset{=}{>} 0$; four distinct real intersections.
3. $R > 0$, but not at once, $s_2 > 0$ and $s_1 s_3 \overset{=}{>} 0$; four distinct imaginary intersections.

$$R = 0, \text{ but not } \frac{3\Delta'}{\Theta'} = \frac{\Theta'}{\Theta} = \frac{\Theta}{3\Delta}.$$

4. $s_1 \overset{=}{>} 0$, $s_2 \overset{=}{>} 0$ $[s_3 \overset{=}{>} 0]$; two coincident real and two distinct real intersections.
5. Not at once $s_1 \overset{=}{>} 0$ and $s_2 \overset{=}{>} 0$, nor $s_2 \equiv 0$, $[s_3 \overset{=}{>} 0]$; two coincident real and two distinct imaginary intersections.
6. $s_1 \overset{=}{<} 0$, $s_2 \equiv 0$ $[s_3 \equiv 0]$, and either $L < 0$ or $M < 0$ or $N < 0$; two distinct pairs of coincident real intersections.
7. $s_1 \overset{=}{<} 0$, $s_2 \equiv 0$ $[s_3 \equiv 0]$, and either $L > 0$ or $M > 0$ or $N > 0$; two distinct conjugate pairs of coincident imaginary intersections.

$$\frac{3\Delta'}{\Theta'} = \frac{\Theta'}{\Theta} = \frac{\Theta}{3\Delta}, \text{ but not } \Delta = \Theta = \Theta' = \Delta' = 0.$$

8. $s_1 \overset{=}{>} 0$ $[s_2 \overset{=}{>} 0,\ s_3 \overset{=}{>} 0]$; three coincident and one isolated intersections, all real.
9. $s_1 \equiv 0$ $[s_2 \equiv 0,\ s_3 \equiv 0]$; four coincident real intersections.

$$\Delta = \Theta = \Theta' = \Delta' = 0, \text{ but not } \frac{a'}{a} = \frac{b'}{b} = \frac{c'}{c} = \frac{f'}{f} = \frac{g'}{g} = \frac{h'}{h}.$$

10. $\mathbf{F} \underset{>}{=} 0$; straight line common, second line of each pair distinct from the other and from the common line, double points distinct.

11. $\mathbf{F} \equiv 0$, either $A < 0$ or $B < 0$ or $C < 0$, either $A' < 0$ or $B' < 0$ or $C' < 0$; straight line common, second line of each pair distinct from the other and from the common line, double points coincident.

12. $\left.\begin{array}{l} \mathbf{F} \equiv 0,\ A = B = C = 0, \text{ either } A' < 0 \text{ or } B' < 0 \text{ or } C' < 0; \\ \text{or } \mathbf{F} \equiv 0,\ A' = B' = C' = 0, \text{ either } A < 0 \text{ or } B < 0 \text{ or } C < 0; \end{array}\right\}$
straight line common which is a double line of one pair and single line of the other pair.

13. $\frac{a'}{a} = \frac{b'}{b} = \frac{c'}{c} = \frac{f'}{f} = \frac{g'}{g} = \frac{h'}{h}$; conics coincident.

The conditions enclosed in brackets are superfluous.

The cases for which $R = 0$ but not $\frac{3\Delta'}{\Theta'} = \frac{\Theta'}{\Theta} = \frac{\Theta}{3\Delta}$ can be distinguished from each other also without the use of s_1, s_2, s_3. In these cases $\Delta_{\lambda\mu} = 0$ has a double root $\frac{\lambda_1}{\mu_1}$ and a single root $\frac{\lambda_2}{\mu_2}$. For these roots we have

$$\frac{\lambda_1}{\mu_1} = -\frac{2(3\Delta'\Theta - \Theta'^2)}{9\Delta\Delta' - \Theta\Theta'} = -\frac{9\Delta\Delta' - \Theta\Theta'}{2(3\Delta\Theta' - \Theta^2)},\quad \frac{\lambda_1^2}{\mu_1^2} = \frac{3\Delta'\Theta - \Theta'^2}{3\Delta\Theta' - \Theta^2},$$

$$\frac{\lambda_2}{\mu_2} = -\frac{\Delta'}{\Delta}\frac{\mu_1^2}{\lambda_1^2} = -\frac{\Delta'(3\Delta\Theta' - \Theta^2)}{\Delta(3\Delta'\Theta - \Theta'^2)},$$

and $\lambda_1 U + \mu_1 V = 0$ may be written

$$(9\Delta\Delta' - \Theta\Theta')U - 2(3\Delta\Theta' - \Theta^2)V = 0 \text{ or } 2(3\Delta'\Theta - \Theta'^2)U - (9\Delta\Delta' - \Theta\Theta')V = 0,$$

the nature of which is determined by the principal first minors of its determinant, which are, by virtue of the equation $R = 0$,

$$2(3\Delta\Theta' - \Theta^2)I,\ 2(3\Delta\Theta' - \Theta^2)J,\ 2(3\Delta\Theta' - \Theta^2)K$$

or

$$2(3\Delta'\Theta - \Theta'^2)I,\ 2(3\Delta'\Theta - \Theta'^2)J,\ 2(3\Delta'\Theta - \Theta'^2)K,$$

where

$$I = 2(3\Delta'\Theta - \Theta'^2)A - (9\Delta\Delta' - \Theta\Theta')(bc' + cb' - 2ff') + 2(3\Delta\Theta' - \Theta^2)A',$$

$$J = 2(3\Delta'\Theta - \Theta'^2)B - (9\Delta\Delta' - \Theta\Theta')(ca' + ac' - 2ff') + 2(3\Delta\Theta' - \Theta^2)B',$$

$$K = 2(3\Delta'\Theta - \Theta'^2)C - (9\Delta\Delta' - \Theta\Theta')(ab' + ba' - 2hh') + 2(3\Delta\Theta' - \Theta^2)C';$$

namely, the first form of the principal minors must be employed if $\Delta' = \Theta' = 0$, the second if $\Delta = \Theta = 0$, but otherwise either.

The criteria for cases 4–7 may then be stated thus:

4. Some one of the quantities I, J, K has the same sign as $3\Delta\Theta' - \Theta^2$ or $3\Delta'\Theta - \Theta'^2$.

5 Some one of the quantities I, J, K has a different sign from $3\Delta\Theta' - \Theta^2$ or $3\Delta'\Theta - \Theta'^2$.

6. $I = J = K = 0$ and either $L < 0$ or $M < 0$ or $N < 0$.

7. $I = J = K = 0$ and either $L > 0$ or $M > 0$ or $N > 0$.

In all of these criteria a vanishing quantity is regarded as having no sign.

The cases in which $\frac{3\Delta'}{\Theta'} = \frac{\Theta'}{\Theta} = \frac{\Theta}{3\Delta}$ but not $\frac{a'}{a} = \frac{b'}{b} = \frac{c'}{c} = \frac{f'}{f} = \frac{g'}{g} = \frac{h'}{h}$ may also be distinguished thus:

8. $\mathbf{F}$ not $\equiv 0$; 9. $\mathbf{F} \equiv 0$:

Or again by the nature of $\sqrt[3]{\Delta'}\,U - \sqrt[3]{\Delta}\,V = 0$, *i. e.* whether it consists of two distinct or two coincident lines, *i. e.* if

$$I' = \sqrt[3]{\Delta'^2}\,A - 2\sqrt[3]{\Delta\Delta'}\,(bc' + cb' - 2ff') + \sqrt[3]{\Delta^2}\,A',$$
$$J' = \sqrt[3]{\Delta'^2}\,B - 2\sqrt[3]{\Delta\Delta'}\,(ca' + ac' - 2gg') + \sqrt[3]{\Delta^2}\,B',$$
$$K' = \sqrt[3]{\Delta'^2}\,C - 2\sqrt[3]{\Delta\Delta'}\,(ab' + ba' - 2hh') + \sqrt[3]{\Delta^2}\,C',$$

then the criteria are:

8. $I' < 0$ or $J' < 0$ or $K' < 0$;

9. $I' = J' = K' = 0$.

V. The nature of the intersection of a quadric surface U and a straight line given as the intersections of two planes P and P', where

$$U \equiv ax^2 + by^2 + cz^2 + dw^2 + 2fyz + 2gzx + 2hxy + 2lxw + 2myw + 2nzw,$$
$$P \equiv \alpha x + \beta y + \gamma z + \delta w, \quad P' \equiv \alpha' x + \beta' y + \gamma' z + \delta' z,$$

is determined by the nature of the roots of the quadratic obtained by eliminating two of the variables from the equations $U = 0$, $P = 0$, $P' = 0$, but a more symmetrical method may be employed. Let x_1, y_1, z_1, w_1 and x_2, y_2, z_2, w_2 be the coordinates of any two different points on the line $P = 0$, $P' = 0$, then the coordinates of any point of this line will be

$$\sigma x_1 + \tau x_2, \quad \sigma y_1 + \tau y_2, \quad \sigma z_1 + \tau z_2, \quad \sigma w_1 + \tau w_2,$$

and for the intersections of the line with the quadric,

$$\sigma^2 U_{11} + 2\sigma\tau U_{12} + \tau^2 U_{22} = 0,$$

where U_{11} and U_{22} are the results of substituting in U the coordinates x_1, y_1, z_1, w_1 and x_2, y_2, z_2, w_2 respectively, and

$$U_{12} \equiv \frac{1}{2}\left(x_1 \frac{\partial U_2}{\partial x_2} + y_1 \frac{\partial U_2}{\partial y_2} + z_1 \frac{\partial U_2}{\partial z_2} + w_1 \frac{\partial U_2}{\partial w_2}\right)$$
$$\equiv \frac{1}{2}\left(x_2 \frac{\partial U_1}{\partial x_1} + y_2 \frac{\partial U_1}{\partial y_1} + z_2 \frac{\partial U_1}{\partial z_1} + w_2 \frac{\partial U_1}{\partial w_1}\right).$$

The nature of the intersections, as is evident from I., depends upon the value of $U_{11}U_{22}-U_{12}^2$, which contains the coordinates only in determinants of the second order; writing for convenience

$$y_1z_2-y_2z_1=(y_1z_2),\ \text{etc.},\quad \beta\gamma'-\beta'\gamma=(\beta\gamma'),\ \text{etc.},$$

we have*

$$\frac{(y_1z_2)}{(\alpha\delta')}=\frac{(z_1x_2)}{(\beta\delta')}=\frac{(x_1y_2)}{(\gamma\delta')}=\frac{(x_1w_2)}{(\beta\gamma')}=\frac{(y_1w_2)}{(\gamma\alpha')}=\frac{(z_1w_2)}{(\alpha\beta')}$$

$=\rho$, say, where ρ is a non-evanescent real quantity. Now

$$U_{11}U_{22}-U_{12}^2=\tfrac{1}{4}\begin{vmatrix} x_1\dfrac{\partial U_1}{\partial x_1}+y_1\dfrac{\partial U_1}{\partial y_1}+z_1\dfrac{\partial U_1}{\partial z_1}+w_1\dfrac{\partial U_1}{\partial w_1}, & x_2\dfrac{\partial U_1}{\partial x_1}+y_2\dfrac{\partial U_1}{\partial y_1}+z_2\dfrac{\partial U_1}{\partial z_1}+w_2\dfrac{\partial U_1}{\partial w_1} \\ x_1\dfrac{\partial U_2}{\partial x_2}+y_1\dfrac{\partial U_2}{\partial y_2}+z_1\dfrac{\partial U_2}{\partial z_2}+w_1\dfrac{\partial U_2}{\partial w_2}, & x_2\dfrac{\partial U_2}{\partial x_2}+y_2\dfrac{\partial U_2}{\partial y_2}+z_2\dfrac{\partial U_2}{\partial z_2}+w_2\dfrac{\partial U_2}{\partial w_2}\end{vmatrix}$$

$$=\tfrac{1}{4}\Big[(y_1z_2)\frac{\partial(U_1,U_2)}{\partial(y_1,z_2)}+(z_1x_2)\frac{\partial(U_1,U_2)}{\partial(z_1,x_2)}+(x_1y_2)\frac{\partial(U_1,U_2)}{\partial(x_1,y_2)}+(x_1w_2)\frac{\partial(U_1,U_2)}{\partial(x_1,w_2)}$$

$$+(y_1w_2)\frac{\partial(U_1,U_2)}{\partial(y_1,w_2)}+(z_1w_2)\frac{\partial(U_1,U_2)}{\partial(z_1,w_2)}\Big],$$

where

$$\frac{\partial(U_1,U_2)}{\partial(y_1,z_2)}=\begin{vmatrix}\dfrac{\partial U_1}{\partial y_1}, & \dfrac{\partial U_1}{\partial z_1} \\ \dfrac{\partial U_2}{\partial y_2}, & \dfrac{\partial U_2}{\partial z_2}\end{vmatrix},\ \text{etc.}$$

i. e.

$$\tfrac{1}{4}\frac{\partial(U_1,U_2)}{\partial(y_1,z_2)}=\begin{vmatrix} hx_1+by_1+fz_1+mw_1, & gx_1+fy_1+cz_1+nw_1 \\ hx_2+by_2+fz_2+mw_2, & gx_2+fy_2+cz_2+nw_2\end{vmatrix}$$

$$=(bc-f^2)(y_1z_2)+(fg-ch)(z_1x_2)+(fh-bg)(x_1y_2)+(hn-gm)(x_1w_2)$$
$$+(bn-fm)(y_1w_2)+(fn-cm)(z_1w_2)$$

$$=\rho\,[(bc-f^2)(\alpha\delta')+(fg-ch)(\beta\delta')+(fh-bg)(\gamma\delta')+(hn-gm)(\beta\gamma')$$
$$+(bn-fm)(\gamma\alpha')+(fn-cm)(\alpha\beta')]$$

$$=\rho\begin{vmatrix} h & g & \alpha & \alpha' \\ b & f & \beta & \beta' \\ f & c & \gamma & \gamma' \\ m & n & \delta & \delta'\end{vmatrix},\ \text{etc.}$$

and substituting the values of (y_1z_2), etc., and $\tfrac{1}{4}\dfrac{\partial(U_1,U_2)}{\partial(y_1,z_2)}$, etc., we find

$$U_{11}U_{22}-U_{12}^2=\rho^2\Big[(\alpha\delta')\begin{vmatrix} h & g & \alpha & \alpha' \\ b & f & \beta & \beta' \\ f & c & \gamma & \gamma' \\ m & n & \delta & \delta'\end{vmatrix}-(\beta\delta')\begin{vmatrix} a & g & \alpha & \alpha' \\ h & f & \beta & \beta' \\ g & c & \gamma & \gamma' \\ l & n & \delta & \delta'\end{vmatrix}$$

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* Salmon's Geometry of Three Dimensions, § 51.

$$+(\gamma\delta')\begin{vmatrix} a & h & \alpha & \alpha' \\ h & b & \beta & \beta' \\ g & f & \gamma & \gamma' \\ l & m & \delta & \delta' \end{vmatrix} + (\beta\gamma')\begin{vmatrix} a & l & \alpha & \alpha' \\ h & m & \beta & \beta' \\ g & n & \gamma & \gamma' \\ l & d & \delta & \delta' \end{vmatrix} + (\gamma\alpha')\begin{vmatrix} h & l & \alpha & \alpha' \\ b & m & \beta & \beta' \\ f & n & \gamma & \gamma' \\ m & d & \delta & \delta' \end{vmatrix}$$

$$+(\alpha\beta')\begin{vmatrix} g & l & \alpha & \alpha' \\ f & m & \beta & \beta' \\ c & n & \gamma & \gamma' \\ n & d & \delta & \delta' \end{vmatrix} \Bigg] = \rho^2 \Delta_{PP'},\ \text{where } \Delta_{PP'} = (abcd)_{PP'} = \begin{vmatrix} a & h & g & l & \alpha & \alpha' \\ h & b & f & m & \beta & \beta' \\ g & f & c & n & \gamma & \gamma' \\ l & m & n & d & \delta & \delta' \\ \alpha & \beta & \gamma & \delta & 0 & 0 \\ \alpha' & \beta' & \gamma' & \delta' & 0 & 0 \end{vmatrix},$$

i. e. Δ, the determinant of U, bordered by the coefficients of P and P'; so that $U_{11}U_{22} - U_{12}^2$ and $\Delta_{PP'}$ have always the same sign when not evanescent, and vanish together.

The intersections of U and the line PP' are then real and distinct, real and coincident, or conjugate imaginary according as $\Delta_{PP'} <, =,$ or > 0.

The condition that the line PP' touches U is then $\Delta_{PP'} = 0$.*

VI. The nature of the intersection of a quadric surface U with a plane P, defined as in V., can be determined by eliminating one variable and examining the nature of the conic represented by the resultant. Eliminating w the resultant is

$$\begin{aligned}&(a\delta^2 - 2l\alpha\delta + d\alpha^2)x^2 + (b\delta^2 - 2m\beta\delta + d\beta^2)y^2 + (c\delta^2 - 2n\gamma\delta + d\gamma^2)z^2 \\ &+ 2(f\delta^2 - m\gamma\delta - n\beta\delta + d\beta\gamma)yz + 2(g\delta^2 - l\gamma\delta - n\alpha\delta + d\alpha\gamma)zx \\ &+ 2(h\delta^2 - l\beta\delta - m\alpha\delta + d\alpha\beta)xy = 0,\end{aligned}$$

the determinant of which is

$$\begin{vmatrix} a\delta^2 - 2l\alpha\delta + d\alpha^2, & h\delta^2 - l\beta\delta - m\alpha\delta + d\alpha\beta, & g\delta^2 - l\gamma\delta - n\alpha\delta + d\alpha\gamma \\ h\delta^2 - l\beta\delta - m\alpha\delta + d\alpha\beta, & b\delta^2 - 2m\beta\delta + d\beta^2, & f\delta^2 - m\gamma\delta - n\beta\delta + d\beta\gamma \\ g\delta^2 - l\gamma\delta - n\alpha\delta + d\alpha\gamma, & f\delta^2 - m\gamma\delta - n\beta\delta + d\beta\gamma, & c\delta^2 - 2n\gamma\delta + d\gamma^2 \end{vmatrix}$$

$$= -\delta^4\Delta_P,$$

where

$$\Delta_P = (abcd)_P = \begin{vmatrix} a & h & g & l & \alpha \\ h & b & f & m & \beta \\ g & f & c & n & \gamma \\ l & m & n & d & \delta \\ \alpha & \beta & \gamma & \delta & 0 \end{vmatrix}$$

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* Salmon's Geometry of Three Dimensions, § 80.

[namely any determinant with the suffix P is bordered by the coefficients of P]; the principal first minors are $-\delta^2(bcd)_P$, $-\delta^2(acd)_P$, $-\delta^2(abd)_P$, where

$$(bcd)_P = \begin{vmatrix} b & f & m & \beta \\ f & c & n & \gamma \\ m & n & d & \delta \\ \beta & \gamma & \delta & 0 \end{vmatrix}, \quad (acd)_P = \begin{vmatrix} a & g & l & \alpha \\ g & c & n & \gamma \\ l & n & d & \delta \\ \alpha & \gamma & \delta & 0 \end{vmatrix}, \quad (abd)_P = \begin{vmatrix} a & h & l & \alpha \\ h & b & m & \beta \\ l & m & d & \delta \\ \alpha & \beta & \delta & 0 \end{vmatrix},$$

and the principal constituents are $-(ad)_P$, $-(bd)_P$, $-(cd)_P$, where

$$(ad)_P = \begin{vmatrix} a & l & \alpha \\ l & d & \delta \\ \alpha & \delta & 0 \end{vmatrix}, \quad (bd)_P = \begin{vmatrix} b & m & \beta \\ m & d & \delta \\ \beta & \delta & 0 \end{vmatrix}, \quad (cd)_P = \begin{vmatrix} c & n & \gamma \\ n & d & \delta \\ \gamma & \delta & 0 \end{vmatrix}.$$

The results of eliminating x, y, or z, instead of w, from $U = 0$ and $P = 0$ are evident from the above; then by III., if we introduce also

$$(abc)_P = \begin{vmatrix} a & h & g & \alpha \\ h & b & f & \beta \\ g & f & c & \gamma \\ \alpha & \beta & \gamma & 0 \end{vmatrix}, \quad (bc)_P = \begin{vmatrix} b & f & \beta \\ f & c & \gamma \\ \beta & \gamma & 0 \end{vmatrix}, \quad (ac)_P = \begin{vmatrix} a & g & \alpha \\ g & c & \gamma \\ \alpha & \gamma & 0 \end{vmatrix}, \quad (ab)_P = \begin{vmatrix} a & h & \alpha \\ h & b & \beta \\ \alpha & \beta & 0 \end{vmatrix},$$

the intersection of P with U will be:

an imaginary true conic, if $(ab)_P(abcd)_P > 0$ and $(abc)_P < 0$;

a real true conic, if $(ab)_P(abcd)_P \overset{=}{<} 0$ or $(abc)_P \overset{=}{>} 0$, but $(abcd)_P \lessgtr 0$;

a pair of conjugate imaginary lines, if $(abcd)_P = 0$ and either $(bcd)_P < 0$ or $(acd)_P < 0$ or $(abd)_P < 0$ or $(abc)_P < 0$;

a pair of distinct real lines, if $(abcd)_P = 0$ and either $(bcd)_P > 0$ or $(acd)_P > 0$ or $(abd)_P > 0$ or $(abc)_P > 0$;

a pair of coincident real lines, if $(abcd)_P = 0$, and $(bcd)_P = (acd)_P = (abd)_P = (abc)_P = 0$.

The condition that P touches U is then $(abcd)_P = 0$.*

If $(ab)_P\Delta_P > 0$ and $(abc)_P < 0$, then also $(ac)_P\Delta_P > 0$, $(ad)_P\Delta_P > 0$, $(bc)_P\Delta_P > 0$, $(bd)_P\Delta_P > 0$, $(cd)_P\Delta_P > 0$, $(bcd)_P < 0$, $(acd)_P < 0$, and $(abd)_P < 0$; and if $\Delta_P = 0$, then those of the determinants $(bcd)_P$, $(acd)_P$, $(abd)_P$, and $(abc)_P$, which do not vanish have the same sign.

VII. The nature of a quadric U is determined by a method analogous to that of III., *i. e.* by considering the nature of the intersections of U with the planes

$$x + \rho w = 0,\ y + \rho w = 0,\ z + \rho w = 0,\ y + \rho z = 0,\ x + \rho z = 0,\ x + \rho y = 0$$

for all real values of ρ. Namely, in order to avoid the consequences of a special relation of U to the tetrahedron of reference, it is necessary to consider the intersections of at least five of these pencils of planes. The results for all these pencils can be obtained from those for any one by an interchange of letters. Let

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* Salmon's Geometry of Three Dimensions, § 79.

$$U \equiv ax^2 + by^2 + cz^2 + dw^2 + 2fyz + 2gzx + 2hxy + 2lxw + 2myw + 2nzw,$$

$$\Delta = (abcd) = \begin{vmatrix} a & h & g & l \\ h & b & f & m \\ g & f & c & n \\ l & m & n & d \end{vmatrix}, \quad A = (bcd) = \begin{vmatrix} b & f & m \\ f & c & n \\ m & n & d \end{vmatrix},$$

$$B = (acd) = \begin{vmatrix} a & g & l \\ g & c & n \\ l & n & d \end{vmatrix}, \quad C = (abd) = \begin{vmatrix} a & h & l \\ h & b & m \\ l & m & d \end{vmatrix}, \quad D = (abc) = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix},$$

$$F = -\begin{vmatrix} a & h & l \\ g & f & n \\ l & m & d \end{vmatrix}, \quad L = -\begin{vmatrix} h & b & f \\ g & f & c \\ l & m & n \end{vmatrix}, \text{ etc.,}$$

$$(ad) = \begin{vmatrix} a & l \\ l & d \end{vmatrix}, \ (bd) = \begin{vmatrix} b & m \\ m & d \end{vmatrix}, \ (cd) = \begin{vmatrix} c & n \\ n & d \end{vmatrix}, \ (bc) = \begin{vmatrix} b & f \\ f & c \end{vmatrix}, \ (ac) = \begin{vmatrix} a & g \\ g & c \end{vmatrix}, \ (ab) = \begin{vmatrix} a & h \\ h & b \end{vmatrix}.$$

The nature of the intersection of U with the plane $x + \rho w$ is determined by the criteria of VI. We have here

$P \equiv x + \rho w$, $(abcd)_P = -(A + 2\rho L + \rho^2 D)$, $(bcd)_P = -\rho^2 (bc)$, $(abc)_P = -(bc)$,
$-(acd)_P = (cd) + 2\rho(gn - cl) + \rho^2 (ac)$,
$-(abd)_P = (bd) + 2\rho(hm - bl) + \rho^2 (ab)$, $(ab)_P = -b$.

1. The intersection with every real plane $x + \rho w$ is then an imaginary true conic if $b(A + 2\rho L + \rho^2 D) > 0$ for every real value of ρ and $(bc) > 0$. Then $AD - L^2 = (bc)\Delta > 0$, so that neither A nor D vanishes, and $bD > 0$. The conditions are therefore

$$(abcd) > 0, \ b(abc) > 0, \ (bc) > 0.$$

2. The intersection is an imaginary true conic for every real plane $x + \rho w$ but one, and a pair of imaginary lines for this one, if $b(A + 2\rho L + \rho^2 D) \geqq 0$ for every real value of ρ and $(bc) > 0$. Then $AD - L^2 = (bc)\Delta = 0$, but neither A nor D vanishes, and $bD > 0$. The conditions are therefore

$$(abcd) = 0, \ b(abc) > 0, \ (bc) > 0.$$

3. The intersection is a real true conic for every real plane $x + \rho w$ if $b(A + 2\rho L + \rho^2 D) \leqq 0$ for every real value of ρ or $(bc) \leqq 0$, but not $A + 2\rho L + \rho^2 D = 0$ for any real value of ρ. Then $AD - L^2 = (bc)\Delta > 0$, so that neither A nor D vanishes; also, if $b(A + 2\rho L + \rho^2 D) \leqq 0$, then $bD \leqq 0$. There are two

possible sets of conditions:

either $(abcd) > 0, \quad b(abc) \overset{=}{<} 0, \quad (bc) > 0,$

or $(abcd) < 0, \quad bc < 0.$

4. The intersection is a real true conic for every real plane $x + \rho w$ but one, say $P_1 \equiv x + \rho_1 w$, and a pair of distinct real lines for this one, if $A + 2\rho L + \rho^2 D = 0$ only for $\rho = \rho_1$, and either $b(A + 2\rho L + \rho^2 D) \overset{=}{<} 0$ for every real value of ρ and $(bc) = 0$ and either $-(abd)_{P_1} < 0$ or $-(acd)_{P_1} < 0$, or $(bc) < 0$. Then $AD - L^2 = (bc)\Delta = 0$. If $(bc) = 0$ it is found that $\rho_1 = -\dfrac{A}{L} = -\dfrac{L}{D} = \dfrac{cm - fn}{ch - fg}$, whence*

$$-(ch - fg)^2 (acd)_{P_1} = (ch - fg)^2 [(cd) + 2\rho_1 (gn - cl) + \rho_1^2 (ac)] = -c^2 \Delta,$$

$$-(ch - fg)^2 (abd)_{P_1} = (ch - fg)^2 [(bd) + 2\rho_1 (hm - bl) + \rho_1^2 (ab)] = -b^2 \Delta,$$

thus $\Delta > 0$;† also, since $b(A + 2\rho L + \rho^2 D) \overset{=}{<} 0$, $bA \overset{=}{<} 0$ and $bD \overset{=}{<} 0$, but not at once, $A = D = 0$, since $A + 2\rho L + \rho^2 D = 0$ only for $\rho = \rho_1$. There are two possible sets of conditions: either

$(abcd) > 0, \quad b(abc) \overset{=}{<} 0, \quad b(bcd) \overset{=}{<} 0,$ but not at once $(abc) = (bcd) = 0, \quad (bc) = 0;$

or $(abcd) = 0, \quad (bc) < 0,$ but not at once $(abc) = (bcd) = 0.$

5. The intersection is a real true conic for every real plane $x + \rho w$ but one, say $P_1 \equiv x + \rho_1 w$, and a pair of imaginary lines for this one, if $A + 2\rho L + \rho^2 D = 0$ only for $\rho = \rho_1$, and either $b(A + 2\rho L + \rho^2 D) \overset{=}{<} 0$ for every real value of ρ and $(bc) = 0$ and either $-(abd)_{P_1} > 0$ or $-(acd)_{P_1} > 0$, or $(bc) > 0$. Then $AD - L^2 = (bc)\Delta = 0$. If $(bc) = 0$, we have, as in 4, $\rho_1 = \dfrac{cm - fn}{ch - fg}$, $-(ch - fg)^2 (acd)_{P_1} = -c^2 \Delta$, $-(ch - fg)^2 (abd)_{P_1} = -b^2 \Delta$, $\therefore \Delta < 0$;‡ also, as in 4, $bA \overset{=}{<} 0$ and $bD \overset{=}{<} 0$, but not at once $A = D = 0$. There are two sets of conditions: either

$(abcd) < 0, \; b(abc) \overset{=}{<} 0, \; b(bcd) \overset{=}{<} 0,$ but not at once $(abc) = (bcd) = 0, \; (bc) = 0,$

or $(abcd) = 0, \; b(abc) \overset{=}{<} 0, \; b(bcd) \overset{=}{<} 0,$ but not at once $(abc) = (bcd) = 0, \; (bc) > 0.$

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[*] The cases in which $(bc) = 0$ and either $c = 0$, or $b = 0$, or $c \lessgtr 0$ and $ch - fg = 0$ require special consideration, but they lead to the same conditions as the general case.

If $c = 0$, then $f = 0$, $\rho_1 = \frac{n}{g}$, $(acd)_{P_1} = 0$, $-(abd)_{P_1} = -\frac{\Delta}{g^2}$, $\therefore \Delta > 0$, $A = -bn^2$, $D = -bg^2$, $bA = -b^2 n^2$, $bD = -b^2 g^2$.

If $b = 0$, we have conditions analogous to those for $c = 0$.

If $c \lessgtr 0$ and $ch - fg = 0$, then $b = \frac{f^2}{c}$, $h = \frac{fg}{c}$, $\rho_1 = \infty$, $(ab) = \frac{f^2}{c^2}(ac)$, $-(abd)_{P_1}$ and $-(acd)_{P_1}$ are of the same sign as (ac), $\therefore (ac) < 0$, $\Delta = -\frac{(ac)}{c^2}(fn - cm)^2 > 0$, $A = -\frac{1}{c}(fn - cm)^2$, $D = 0$, $ba = -\frac{f^2}{c^2}(fn - cm)^2 \overset{=}{<} 0$.

[†] It cannot be that $b = c = 0$, for then $A = D = 0$, which is excluded by the next line.

[‡] See also the first footnote to 4.

6. The intersection is a real true conic for every real plane $x + \rho w$ but one, say $P_1 \equiv x + \rho_1 w$, and a pair of coincident real lines for this one, if $A + 2\rho L + \rho^2 D = 0$ only for $\rho = \rho_1$, and $b(A + 2\rho L + \rho^2 D) \overset{=}{<} 0$ for every real value of ρ, $(bc) = 0$, $(abd)_{P_1} = (acd)_{P_1} = 0$. Then $AD - L^2 = (bc)\,\Delta = 0$, $\rho_1 = \dfrac{cm - fn}{ch - fg}$, $-(ch - fg)^2 (acd)_{P_1} = -c^2\Delta$, $-(ch - fg)^2 (abd)_{P_1} = -b^2\Delta$, $\therefore\ \Delta = 0$;* also $bA \overset{=}{<} 0$ and $bD \overset{=}{<} 0$, but not at once $A = D = 0$. The conditions are

$$(abcd) = 0,\ b(abc) \overset{=}{<} 0,\ b(bcd) \overset{=}{<} 0,\ \text{but not at once } (abc) = (bcd) = 0,\ (bc) = 0.$$

7. The intersection is a real true conic for every real plane $x + \rho w$ but two if $A + 2\rho L + \rho^2 D = 0$ for two real values of ρ, and either $b(A + 2\rho L + \rho^2 D) \overset{=}{<} 0$ for every real value of ρ or $(bc) < 0$. Then $AD - L^2 = (bc)\,\Delta < 0$, and hence $(bc) \lessgtr 0$. Since $A + 2\rho L + \rho^2 D$ changes sign for real values of ρ, $b(A + 2\rho L + \rho^2 D) \overset{=}{<} 0$ only if $b = 0$, but then $(bc) < 0$. The conditions are therefore

$$(abcd) > 0,\ (bc) < 0.$$

8. The intersection is an imaginary true conic for an infinite succession of real planes $x + \rho w$, a real true conic for an infinite succession of real planes $x + \rho w$, and a pair of imaginary straight lines for each of the two real planes which separate these successions, if $AD - L^2 = (bc)\Delta < 0$ and $(bc) > 0$, *i. e.* if

$$(abcd) < 0,\ (bc) > 0.$$

9. The intersection is a pair of imaginary lines for every real plane $x + \rho w$ if $A = L = D = 0$ and either $(bc) > 0$, or $(bc) = 0$, $-(acd)_P \overset{=}{>} 0$ and $-(abd)_P \overset{=}{>} 0$ for every real value of ρ, but not at once $(abd)_P = (acd)_P = 0$. Hence, if $(bc) = 0$, $(ac)(cd) - (gn - cl)^2 = cB \overset{=}{>} 0$ and $(ab)(bd) - (hm - bl)^2 = bC \overset{=}{>} 0$, $(ab) > 0$ or $(ac) > 0$ or $(bd) > 0$ or $(cd) > 0$, together with such other conditions as shall ensure the inequality of the values of ρ for which $(abd)_P$ and $(acd)_P$ vanish, if the latter are both perfect squares, *i. e.* neither of them can be a finite (or vanishing) multiple of the other if $bC = cB = 0$. Now if $(bc) = 0$ and $A = D = 0$, we have, if $c \lessgtr 0$, $b = \dfrac{f^2}{c}$, $A = -\dfrac{1}{c}(cm - fn)^2$, $D = -\dfrac{1}{c}(ch - fg)^2$, $\therefore\ m = \dfrac{fn}{c}$, $h = \dfrac{fg}{c}$, $(ab) = \dfrac{f^2}{c^2}(ac)$, $(bd) = \dfrac{f^2}{c^2}(cd)$, $hm - bl^2 = \dfrac{f^2}{c^2}(gn - cl)$, $(abd)_P = \dfrac{f^2}{c^2}(acd)_P$; if $c = 0$, we have $f = 0$, $A = -bn^2$, $D = -bg^2$, $\therefore\ g = n = 0$ (since $b = c = 0$ makes $(ab) \overset{=}{<} 0$, $(ac) \overset{=}{<} 0$, $(bd) \overset{=}{<} 0$, $(cd) \overset{=}{<} 0$), $(ac) = (cd) = 0$, $gn - cl = 0$, $(acd)_P \equiv 0$; in each of these cases one of the quantities $(abd)_P$ and $(acd)_P$ is a finite (or vanishing) multiple of the other, therefore we cannot have at

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*See also the first footnote to 4.

once $bC = cB = 0$, but must have either $bC > 0$ or $cB > 0$. If either of the above sets of conditions are satisfied it will be found that $\Delta = 0$; also, if $A = D = 0$ and either $\Delta = 0$ or $(bc) = 0$, then $L = 0$. The conditions for this species of intersection are then

either $\quad (abcd) = 0,\ (abc) = (bcd) = 0,\ (bc) > 0$;

or $\quad (abcd) = 0,\ (abc) = (bcd) = 0,\ b(abd) > 0$ or $c(acd) > 0,\ (bc) = 0$,
$(ab) > 0$ or $(ac) > 0$ or $(bd) > 0$ or $(cd) > 0$.

10. The intersection is a pair of imaginary lines for every real plane $x + \rho w$ but one, and a pair of coincident real lines for that one, if $A = D = L = 0$, $(bc) = 0$, $-(acd)_P \geqq 0$ and $-(abd)_P \geqq 0$ for every real value of ρ, namely $(abd)_P = (acd)_P = 0$ for one and only one real value of ρ. Then, as is evident from the treatment of the preceding case, the conditions for this species of intersection are

$(abcd) = 0,\ (abc) = (bcd) = 0,\ b(abd) = c(acd) = 0,\ (bc) = 0,\ (ab) > 0$ or $(ac) > 0$ or $(bd) > 0$ or $(cd) > 0$.

11. The intersection is a pair of distinct real lines for every real plane $x + \rho w$, if $A = D = L = 0$ and either $(bc) < 0$, or $(bc) = 0$ and $-(abd)_P \leqq 0$ and $-(acd)_P \leqq 0$ for every real value of ρ, but not at once $(abd)_P = (acd)_P = 0$. If $(bc) = 0$, the conditions are the same as in 9, excepting that here $(ab) < 0$ or $(ac) < 0$ or $(bd) < 0$ or $(cd) < 0$, and hence $b = c = f = 0$ is possible; if then $b = c = f = 0$, $-(acd)_P \equiv -(n - \rho g)^2$, $-(abd)_P \equiv -(m - \rho h)^2$, and we must have $gm - hn \lessgtr 0$, but we also have $\Delta = (gm - hn)^2$, hence $\Delta > 0$. The conditions for this species of intersection are then

either $\quad (abcd) = 0,\ (abc) = (bcd) = 0,\ (bc) < 0$;

or $\quad (abcd) = 0,\ (abc) = (bcd) = 0,\ b(abd) > 0$ or $c(acd) > 0,\ (bc) = 0$,
$(ab) < 0$ or $(ac) < 0$ or $(bd) < 0$ or $(cd) < 0$;

or $\quad (abcd) > 0,\ (abc) = (bcd) = 0,\ (bc) = 0$.

12. The intersection is a pair of distinct real lines for every real plane $x + \rho w$ but one, and a pair of coincident real lines for that one, if $A = D = L = 0$, $(bc) = 0$, $-(abd)_P \leqq 0$ and $-(acd)_P \leqq 0$ for every real value of ρ, namely $(abd)_P = (acd)_P = 0$ for one and only one real value of ρ. The conditions are then

$(abcd) = 0,\ (abc) = (bcd) = 0,\ b(abd) = c(acd) = 0,\ (bc) = 0,\ (ab) < 0$ or $(ac) < 0$ or $(bd) < 0$ or $(cd) < 0$.

13. The intersection is a pair of imaginary lines for an infinite succession of real planes $x + \rho w$, a pair of distinct real lines for an infinite succession of real planes $x + \rho w$, and a pair of coincident real lines for each of the two real

planes which separate these successions, if $A = D = L = 0$, $(bc) = 0$, $bC < 0$ or $cB < 0$. Then, as in 9, since b and c cannot both vanish, $\Delta = 0$. Hence the conditions are

$$(abcd) = 0,\ (abc) = (bcd) = 0,\ b(abd) < 0 \text{ or } c(acd) < 0,\ (bc) = 0.$$

14. The intersection is a pair of coincident real lines for every real plane $x + \rho w$, if $A = D = L = 0$, $(bc) = 0$, $b(abd) = c(acd) = 0$, $(ab) = (ac) = (bd) = (cd) = 0$. Then $\Delta = 0$, and the conditions are

$$(abcd) = 0,\ (abc) = (bcd) = 0,\ b(abd) = c(acd) = 0,\ (bc) = 0,\ (ab) = (ac) \\ = (bd) = (cd) = 0.$$

These results tabulated are:

$(abcd)$	(bc)	$b(abc)$ $b(bcd)$	(abc) (bcd)	$b(abd)$ $c(acd)$	(ab), (ac) (bd), (cd)	Species
>0	>0	either >0				1
		either $\overset{=}{<} 0$				3
	<0					7
	$=0$		either $\lessgtr 0$			4
			both $=0$			11
<0	>0					8
	<0					3
	$=0$					5
$=0$	>0	either >0				2
		either $\overset{=}{<} 0$	either $\lessgtr 0$			5
			both $=0$			9
	<0		either $\lessgtr 0$			4
			both $=0$			11
	$=0$		either $\lessgtr 0$			6
			both $=0$	either >0	either >0	9
					either <0	11
				either <0		13
				both $=0$	either >0	10
					either <0	12
					all $=0$	14

The interchanges of letters by which this table is made to give the conditions for the different species of intersections with planes through either edge of the tetrahedron of reference are evident.

There are the following eight absolute classes of quadric surfaces (surfaces of the second order given by real equations in point-coordinates):

imaginary true quadrics,
convex quadrics (real true quadrics with imaginary generators),
real quadric scrolls (with real generators),
imaginary quadric cones (with real vertices),
real quadric cones,
pairs of imaginary planes (with real double edges),
pairs of distinct real planes,
pairs of coincident real planes.

For an imaginary true quadric the intersections are of the species 1 for each edge of the tetrahedron, and the necessary and sufficient conditions for this class of quadrics may be written

$$(abcd) > 0, \quad (ab) > 0, \quad a(abc) > 0.$$

A convex quadric may be cut by either edge of the tetrahedron in distinct real points, in coincident real points (case of tangency), or in imaginary points; the intersections by a plane rotating about the edge are of the species 3, 5 or 8, which concur only if $(abcd) < 0$, which is therefore the condition for this class.

A real quadric scroll may be cut by either edge in distinct real points, in coincident real points (case of tangency), or in imaginary points, or it may contain the edge as generator; the intersections by the rotating plane may be of the species 7, 4, 3 or 11, which concur only if $(abcd) > 0$, but the other conditions for 1 must not all be satisfied, *i. e.* the conditions for this class are

$$(abcd) > 0, \text{ and either } (ab) \leqq 0 \text{ or } a(abc) \leqq 0.$$

An imaginary quadric cone may be cut by either edge of the tetrahedron in imaginary points or in coincident real points (the edge passing through the vertex of the cone), but not more than three edges can have the latter position; the intersections by the revolving plane are of the species 2 or 9; and, conversely, if the intersections are of the species 2 for either edge, the quadric is evidently an imaginary cone. The conditions are therefore

$$(abcd) = 0, \text{ and either } a(abc) > 0 \text{ and } (ab) > 0, \text{ or } a(abd) > 0 \text{ and } (ab) > 0,$$
$$\text{or } c(acd) > 0 \text{ and } (cd) > 0, \text{ or } c(bcd) > 0 \text{ and } (cd) > 0.$$

A real cone is met by any edge of the tetrahedron in imaginary points, in distinct real points, or in coincident real points so that the edge is tangent to the cone or passes through the vertex, or the cone contains the edge as a generator; the intersections with the revolving plane are of the species 5, 4, 6, 13, 11 or 12, namely, of the species 12 for at most three edges meeting in one point. Now by a real linear transformation which does not alter the value of Δ, any tangent to the cone can be made an edge of the tetrahedron of reference, hence $\Delta = 0$. Then the conditions are evidently

$$(abcd) = 0, \text{ either } (ab) \overset{=}{<} 0, \text{ or } a(abc) \overset{=}{<} 0 \text{ and } a(abd) \overset{=}{<} 0,$$

and either $(cd) \overset{=}{<} 0$, or $c(acd) \overset{=}{<} 0$, and $c(bcd) \overset{=}{<} 0$.

A pair of imaginary planes is met by any edge of the tetrahedron in two imaginary points, or in two coincident real points, or the edge lies entirely in the quadric (is the double line of the pair of planes), but only one edge can have the latter position; the intersections with the revolving plane are of the species 9, 10 or 14, namely, of the species 9 for at least one edge, and of the species 14 for at most one edge. The conditions are therefore

$$(abcd) = 0,\ (bcd) = (acd) = (abd) = (abc) = 0, \text{ and either } (ab) > 0$$
$$\text{or } (ac) > 0 \text{ or } (ad) > 0 \text{ or } (bc) > 0 \text{ or } (bd) > 0 \text{ or } (cd) > 0.$$

A pair of distinct real planes is met by any edge of the tetrahedron in two distinct real points, or in two coincident real points, or the edge lies in one or both of the planes of the pair, and at least one edge must have the first position and at most one edge can have the last position (coincident with the double line of the pair of planes); the intersections with the revolving plane are of the species 11, 12 or 14, namely, of the species 11 for at least one edge, and of the species 14 for at most one edge. By a real linear transformation which does not alter the value of Δ, the double line of the pair of planes can be made an edge of the tetrahedron, hence $\Delta = 0$. The conditions are then

$$(abcd) = 0,\ (bcd) = (acd) = (abd) = (abc) = 0, \text{ and either } (ab) < 0$$
$$\text{or } (ac) < 0 \text{ or } (ad) < 0 \text{ or } (bc) < 0 \text{ or } (bd) < 0 \text{ or } (cd) < 0.$$

A pair of coincident real planes is met by any edge of the tetrahedron in two coincident real points; the intersections with the revolving plane are of the species 14 for every edge. The conditions are therefore

$$(abcd) = 0,\ (bcd) = (acd) = (abd) = (abc) = 0,$$
$$(ab) = (ac) = (ad) = (bc) = (bd) = (cd) = 0.$$

The conditions here given for the different classes are mutually exclusive and therefore not only necessary but sufficient. They can be put into a somewhat simpler form. Any succession of principal minors of Δ (including Δ itself) of which each is a principal first minor of the preceding may be called a *sequence* of principal minors of Δ, and the minors themselves may be denoted by (4), (3), (2), (1) respectively, the number enclosed in parentheses denoting the order of the determinant constituting the minor; thus there is a sequence in which $(4) = (abcd)$, $(3) = (abd)$, $(2) = (bd)$, $(1) = b$; (4) is always $(abcd)$, and every (1) is a principal constituent of $(abcd)$. The conditions for the different classes are then the following:

if $(4) > 0$ and there is a sequence in which $(1)(3) > 0$ and $(2) > 0$, U is an imaginary true quadric;

if $(4) > 0$ and there is no sequence in which $(1)(3) > 0$ and $(2) > 0$, U is a quadric scroll;

if $(4) < 0$, U is a convex quadric;

if $(4) = 0$ and there is a sequence in which $(1)(3) > 0$ and $(2) > 0$, U is an imaginary cone;

if $(4) = 0$ and there is a sequence in which either $(1)(3) \overset{=}{<} 0$ or $(2) \overset{=}{<} 0$, but not $(3) = 0$, U is a real cone;

if $(4) = 0$ and in every sequence $(3) = 0$, and in at least one sequence $(2) > 0$, U is a pair of imaginary planes;

if $(4) = 0$ and in every sequence $(3) = 0$, and in at least one sequence $(2) < 0$, U is a pair of distinct real planes;

if $(4) = 0$ and in every sequence $(3) = 0$ and $(2) = 0$, U is a pair of coincident real planes.

(TO BE CONTINUED.)